Fuel-Optimal Orbital Transfers to Mars

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Interplanetary missions to Mars are generally heavily constrained by limited fuel availability. As such, it is necessary to select launch and landing dates such that the ΔV required for the transfer is not prohibitively large. This research seeks to minimize the ΔV of the interplanetary transfer by formulating an optimization problem with two design variables: Earth departure date and Mars arrival date. A variety of algorithms, including deterministic and stochastic methods, are employed to optimize the objective function. Two solutions are found within the search space, corresponding to Type I and Type II transfers. Finally, these results are compared with a recent mission implementation for verification.

I. Introduction

Launch windows and transfer times are important considerations in the mission design process for any space-faring mission. These variables are generally heavily influenced by the the geometry of the initial and final orbits of the spacecraft. For transfers to Mars, timing considerations become even more crucial, as transfers must be designed with proper phasing in mind such that the spacecraft can properly intercept with the target body. Knowledge of orbital dynamics and Earth-Mars geometry as functions of time can be leveraged to aid in the mission design process and determine optimal transfer properties.

Lambert's algorithm is a well-known method for the orbit determination of a transfer arc given two position vectors and a Time of Flight (TOF) between the start and end states [1] [2]. This algorithm can be applied to the problem of Earth-Mars transfers by uniquely determining the minimum energy transfer orbit given an Earth departure date and Mars arrival date. Once the transfer orbit is known, the excess hyperbolic velocity (known as C3) needed for departure and arrival can be calculated. Making some assumptions on the initial Earth-centered orbit and final Mars-centered orbit, the total ΔV (and therefore total fuel expenditure) can also be easily recovered.

This research seeks to solve an optimization problem to determine the desired launch epoch and transfer time for a fuel-optimal transfer from Low Earth Orbit (LEO) to Mars. This is modeled as a 2D constrained optimization problem where the independent parameters X_1 and X_2 are the transfer start and end epochs, respectively. The objective function $f(X_1, X_2)$ is then the total ΔV required for the transfer. The only constraints on the problem are that the TOF is between 0 and 1.5 years to avoid prohibitively long travel times. By solving Lambert's Problem at each iteration, the dynamics constraints are inherently met with each transfer arc solution, thus allowing the optimization problem to operate without any additional constraints. The solution of this optimization problem gives the desired transfer initiation and termination epochs for a minimum ΔV Earth-Mars transfer.

II. Theory

A. Lambert's Algorithm

Lambert's problem is a well-studied problem concerned with the orbit determination of a transfer trajectory given known initial and final position vectors and a transfer time in between them. Lambert's algorithm represents the solution to this two point boundary value problem and is frequently used in interplanetary mission planning. By assuming a patched conics model, the dynamics of the transfer arc are influenced only by the gravity of the central body (typically the Sun) and are governed by the Fundamental Orbital Differential Equation (FODE):

$$\frac{d^2}{dt^2}\vec{r} + \mu\frac{\vec{r}}{r^3} = 0 \tag{1}$$

In the above, $\vec{r}$ denotes the position vector from the central body to the spacecraft and μ represents the gravitational constant of the central body. Solutions of the FODE take the form of conic sections. For interplanetary trajectories, the initial and final points, $\vec{r}_1$ and $\vec{r}_2$, respectively, are typically known from planetary ephemeris data. Taking in $\vec{r}_1, \vec{r}_2$, and Δt as inputs, Lambert's algorithm leverages the geometry of the problem to solve for the orbital parameters defining the transfer conic.

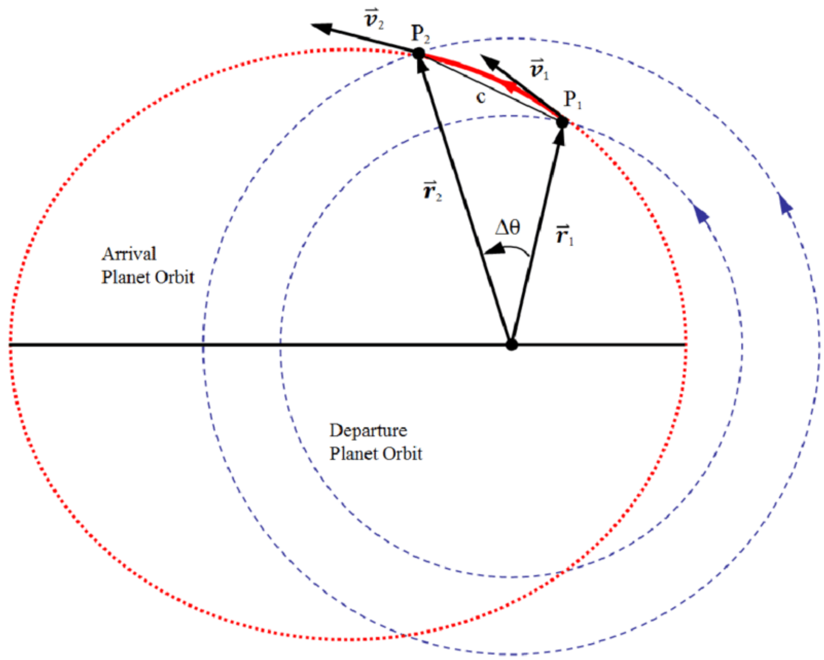

Fig. 1 Geometry of Lambert's Problem [3]

Figure 1 depicts an example case showing the geometry of Lambert's Problem. Here, P_1 is the position of the departure body at time t_1 and P_2 is the position of the arrival planet at time t_2. The solid red line defines the transfer trajectory as solved for by Lambert's Algorithm, with the dashed red line defining the geometry of the full transfer orbit. Note that there are generally two solutions to Lambert's problem for a given $\vec{r}_1, \vec{r}_2$, and Δt, corresponding to Type I and Type II trajectories. Type I trajectories (Ex: Figure 1) are characterized by a transfer angle $\Delta\theta < 180°$, whereas Type II trajectories have $\Delta\theta \geq 180°$.

B. Gradient Descent

Gradient descent is a first-order optimization method that simply uses the negative of the gradient as the descent direction at each iteration. The step size is determined by a hyperparameter α, which can be constant or decay over successive iterations. The update step for the gradient descent algorithm (with normalized gradients and decaying step factor) is as follows:

$$x^{(k+1)} = x^{(k)} - \alpha \frac{g^{(k)}}{||g^{(k)}||} \tag{2}$$

$$\alpha^{(k+1)} = \gamma \alpha^{(k)} \tag{3}$$

In the above, we've defined $g^{(k)} = \nabla f(x^{(k)})$ for convenience. The two hyperparameters of this algorithm are the learning rate α and decay factor γ.

C. Adam

The adaptive moment estimation algorithm, or Adam, is another first-order method that overcomes some of the limitations of gradient descent by adapting learning rates to each parameter and storing momentum-like information.

The update step for an iteration of Adam is given in [4] and also shown here:

$$\text{biased decaying momentum: } v^{(k+1)} = \gamma_v v^{(k)} + (1-\gamma_v) g^{(k)} \tag{4}$$
$$\text{biased decaying sq. gradient: } s^{(k+1)} = \gamma_s s^{(k)} + (1-\gamma_s)(g^{(k)} \odot g^{(k)}) \tag{5}$$
$$\text{corrected decaying momentum: } \hat{v}^{(k+1)} = v^{(k+1)}/(1-\gamma_v^k) \tag{6}$$
$$\text{corrected decaying sq. gradient: } \hat{s}^{(k+1)} = s^{(k+1)}/(1-\gamma_s^k) \tag{7}$$
$$\text{next iterate: } x^{(k+1)} = x^{(k)} - \alpha \hat{v}^{(k+1)}/(\epsilon + \sqrt{\hat{s}^{(k+1)}}) \tag{8}$$

At each iteration of Adam, an exponentially decaying momentum and squared gradient is calculated (Eqs. 4-5). To reduce bias introduced by initializing these values to zero, a bias correction step is introduced (Eqs. 6-7). The corrected momentum and squared gradient values are then used to calculate the next iterate.

D. Cross-Entropy

Both of the aforementioned algorithms are deterministic in nature, and are thus sensitive to initial conditions and more susceptible to locally optimal solutions. The Cross-Entropy method is stochastic in nature, thus benefiting from the use of randomness to better converge to a globally optimal solution. At each iteration, it randomly samples N samples of $f(\vec{X})$ from a proposal distribution parameterized by θ. From this set of samples, a subset of size M of the best performing samples (frequently referred to as "elite" samples) is used to fit the new proposal distribution used in the next iteration. A common choice of proposal distributions is the multivariate Gaussian. When fitting elite samples to the multivariate Gaussian, the maximum likelihood estimate (MLE) is used to fit the parameters $\theta = (\mu, \Sigma)$:

$$\mu^{(k+1)} = \frac{1}{M}\sum_{i=1}^{M} x^{(i)} \tag{9}$$
$$\Sigma^{(k+1)} = \frac{1}{M}\sum_{i=1}^{M} (x^{(i)} - \mu^{(k+1)})(x^{(i)} - \mu^{(k+1)})^T \tag{10}$$

III. Modeling

A. Dynamics

The orbits of Earth and Mars around the Sun were initialized at the J2000 epoch using Keplerian elements data collected by the Jet Propulsion Laboratory [5]. By assuming two-body Keplerian motion, analytical methods could be leveraged to efficiently propagate the orbits of Earth and Mars forward in time with low computational cost. Therefore, Earth's heliocentric position and velocity $\vec{r}_1, \vec{v}_1^-$ and Mars' position and velocity $\vec{r}_2, \vec{v}_2^+$ are known for a given departure epoch t_1 and arrival epoch t_2. The solution to Lambert's problem then gives the heliocentric departure velocity $\vec{v}_1^+$ at t_1 and arrival velocity $\vec{v}_2^-$ at t_2. The magnitude of the hyperbolic excess velocities required for Earth departure and Mars arrival are then simply the L2 norm of the difference in velocities:

$$\text{Earth Departure: } v_{\infty,1} = ||\vec{v}_1^+ - \vec{v}_1^-||_2 \tag{11}$$
$$\text{Mars Arrival: } v_{\infty,2} = ||\vec{v}_2^+ - \vec{v}_2^-||_2 \tag{12}$$

Note that the characteristic energy (C_3), a common metric used to measure the excess specific energy after Earth departure, is simply $C_3 = v_{\infty,1}^2$. Although C_3 gives a sufficient measure of mission cost and is frequently used by mission designers, total ΔV (a more tangible measure of fuel cost) can also be calculated by making a few assumptions on the initial and final orbits. The initial Earth parking orbit is assumed to be a 400 km altitude circular orbit in the same plane as the transfer arc (such that no ΔV is expended for plane changes). Similarly, the final Mars parking orbit is assumed to be a 400 km altitude circular orbit lying in the same plane as the transfer trajectory. With these assumptions, the required ΔV for the transfer is then:

$$\text{Earth Departure: } \Delta V_1 = v_{c,1} + \sqrt{v_{\infty,1}^2 + \frac{2\mu_E}{a_1}} \tag{13}$$

$$\text{Mars Arrival: } \Delta V_2 = v_{c,2} + \sqrt{v_{\infty,2}^2 + \frac{2\mu_M}{a_2}} \tag{14}$$

$$\text{Total: } \Delta V = \Delta V_1 + \Delta V_2 \tag{15}$$

In the above, $(v_{c,1}, v_{c,2})$ are the velocities of the initial and final parking orbits, (a_1, a_2) are the radii of the initial and final parking orbits, and (μ_E, μ_M) are the gravitational parameters of Earth and Mars, respectively. The total ΔV assumes the spacecraft is in an initial Low Earth Orbit (LEO) and ends in Martian orbit, and thus does not include the costs associated with launch/landing.

B. Optimization Problem

A two-dimensional constrained optimization problem was modeled to determine the fuel-optimal transfer to Mars. The Earth departure Modified Julian Date (MJD) and Mars arrival MJD were modeled as the two independent parameters, X_1 and X_2, of the optimization algorithm. The MJD was used as it provides a convenient floating-point representation of a given date/time. Using the methodology outlined in the previous subsection, the ΔV associated with a given launch and arrival MJD can be calculated. Since the goal of this research is to minimize fuel-expenditure (and therefore ΔV), the objective function is simply $f(X_1, X_2) = \Delta V$. The problem was constrained such that the arrival date must occur after the departure date and that the TOF is no greater than 1.5 years (547.5 days). The optimization problem is then formulated as follows:

$$\underset{\vec{X}}{\text{minimize}} \quad f(\vec{X}), \tag{16}$$

$$\text{subject to} \quad X_1 - X_2 \leq 0 \tag{17}$$

$$X_2 - X_1 - 547.5 \leq 0 \tag{18}$$

To better optimize while accounting for these constraints, count and quadratic penalty methods were implemented in the objective function to ensure the optimization routine converged to a feasible solution. With penalty methods, the optimization problem could then be reformulated as follows:

$$\underset{\vec{X}}{\text{minimize}} \quad f(\vec{X}) + \rho_c \sum_{i=1}^{2} (g_i(\vec{X}) > 0) + \rho_q \sum_{i=1}^{2} \max(g_i(\vec{X}), 0)^2 \tag{19}$$

$$\text{subject to} \quad \vec{X} \in \mathbb{R}^2 \tag{20}$$

where $g_i(\vec{X})$ refers to the i^{th} constraint $g(\vec{X}) \leq 0$ (Eqs. 17-18), ρ_c denotes the count penalty magnitude, and ρ_q denotes the quadratic penalty magnitude. The inclusion of both count and quadratic penalties ensures that a clear delineation exists between feasible and infeasible points while also preserving gradient information to guide the solver towards the feasible set.

IV. Results and Discussion

Two first-order methods, gradient descent and Adam, were first implemented to optimize ΔV (using the central difference method to numerically approximate gradients). Figure 2 compares the performance of these two algorithms for three different initial conditions. The paths taken by the optimization routines are overlaid on top of contour plots depicting the ΔV cost for a given departure and arrival date. These contour lines were generated by running Lambert's Algorithm iteratively across the search space to generate what is commonly referred to by mission designers as a "Porkchop Plot" [3]. Porkchop plots are frequently used for interplanetary trajectory planning and in this case allow us to better visualize performance of varying algorithms. The shaded red regions depict infeasible solutions as specified by our TOF constraints. A 2020 launch season was chosen so that results could be compared with recent mission

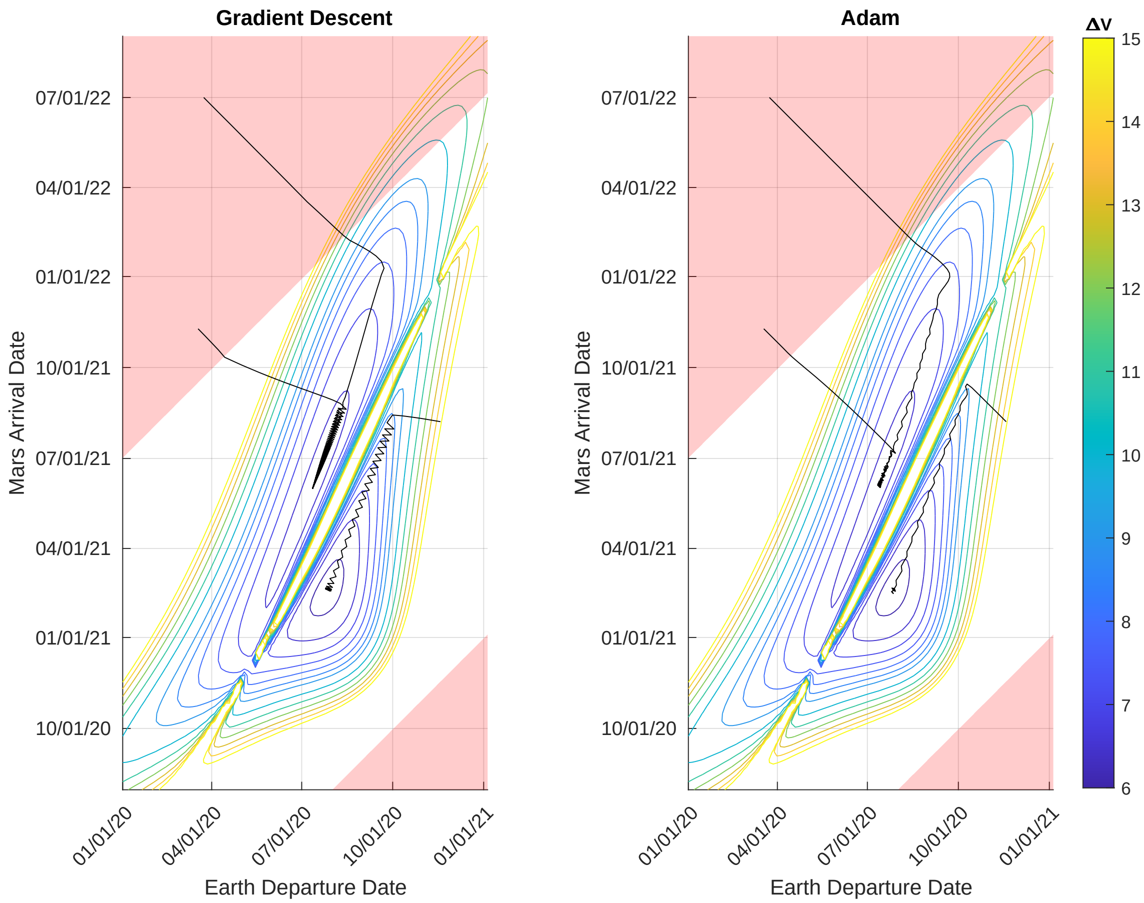

Fig. 2 Comparison of Gradient Descent and Adam Performance

implementations (i.e. the Mars 2020 mission). However, it is known that launch geometry for Martian missions repeats roughly every 2.1 years, therefore this research could be extended to an infinite number of potential launch seasons.

As can be seen by Figure 2, both first-order methods appear to consistently find locally optimal solutions. They also both do a good job of quickly moving towards the feasible space, as influenced by the penalty functions introduced on the objective function. However, even after offline hyperparameter tuning, gradient descent appears to suffer from very choppy behavior due to the geometry of the narrow "valley" it is moving within. Adam appears to better account for these narrow regions by storing past gradient information which influences its future direction of travel, and thus has a smoothing effect on the search path. Additionally, Adam was run for 100 iterations whereas gradient descent was run for 1000, so Adam typically converged in only a fraction of the iterations.

The most noticeable shortcoming of both of these algorithms is that they frequently get stuck in locally optimal solutions depending on the initial condition. As is common in interplanetary launch windows, there appears to be two distinct regions in our search space corresponding to Type I and Type II trajectories, each with its own optimum. Table 1

Table 1 Comparison of Optimal Type I and Type II Transfers

	Type I Transfer	**Type II Transfer**
Earth Departure Date	07/26/2020	07/11/2020
Mars Arrival Date	02/18/2021	05/31/2021
Time of Flight (days)	206.8	324.7
Total ΔV (km/s)	5.849	6.346

summarizes the key differences between the Type I and Type II solutions. As can be seen, for the 2020 launch season Type I trajectories appear to be globally optimal (though this varies between launch seasons). Type II trajectories might also be useful in certain cases depending on other mission considerations (i.e. TOF, program schedule, etc.). Figure 3 depicts the orbital geometry of the optimal Type I and Type II trajectories projected onto the heliocentric ecliptic plane.

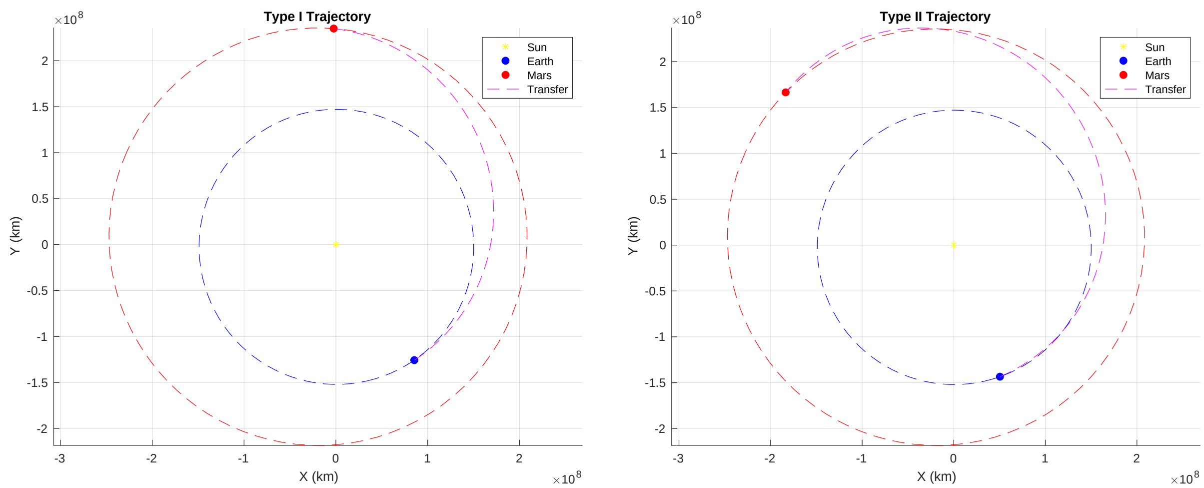

Fig. 3 Orbital Geometry of Optimal Type I and Type II Transfers

Since the deterministic first-order methods are susceptible to the locally optimal (Type II) solution, the Cross-Entropy algorithm was implemented in order to more reliably converge to the globally optimal (Type I) solution.

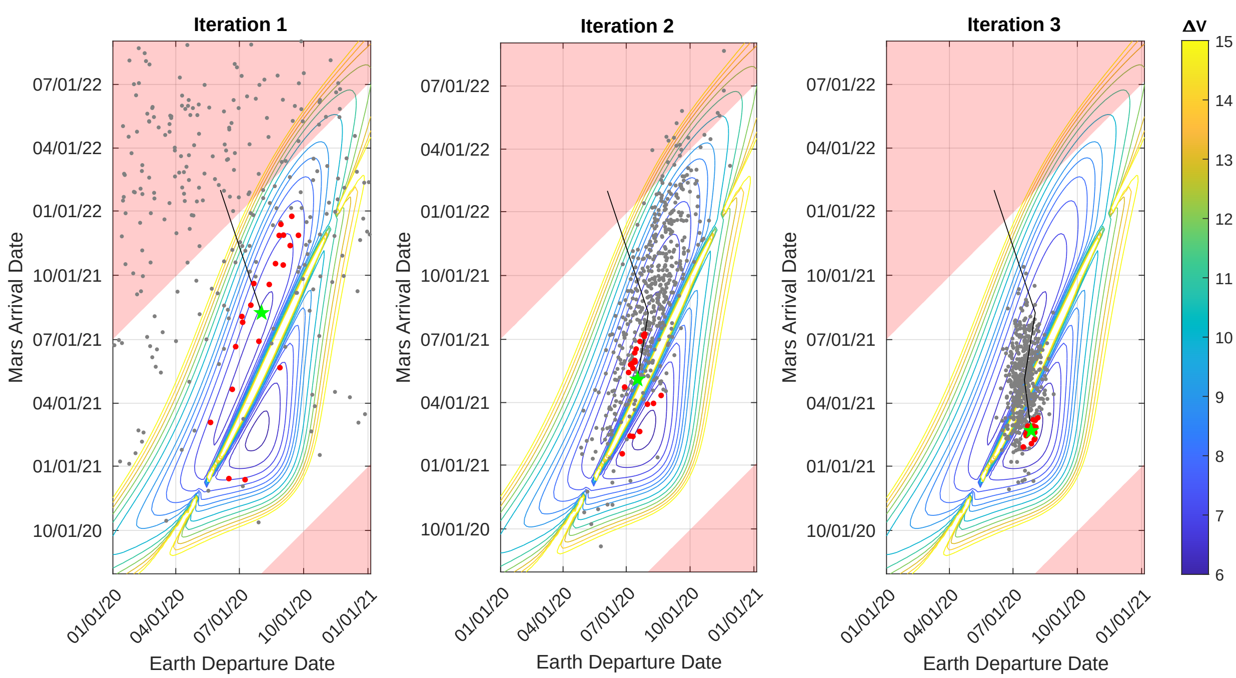

Fig. 4 Three Iterations of the Cross-Entropy Algorithm

The Cross-Entropy method was initialized with a randomly seeded mean and large covariance matrix such that the full search space could be adequately covered. At each iteration, the algorithm randomly sampled 500 points (gray) and selected 20 elite samples (red) based on objective function values. These elite samples were then used to fit a new Gaussian distribution with mean μ (green) and covariance Σ. As can be seen from Figure 4, with this selection of hyperparameters, the algorithm generally converged within three iterations. Additionally, the stochastic nature of this method allowed it to escape the locally optimal Type II solution in favor of the global solution. This robustness to initial conditions is a clear advantage of the Cross-Entropy method over the two previously implemented deterministic algorithms.

The Cross-Entropy algorithm reliably converged to the globally optimal solution as given by the Type I transfer in Table 1. To verify these results, the departure/arrival dates were compared with the launch/landing dates of a recent mission implementation: the Mars 2020 mission. Mars 2020 is an ongoing mission managed by the Jet Propulsion Laboratory to further scientific exploration of Mars. The mission launched the Perseverance rover and Ingenuity helicopter on 07/30/2020 and landed on the Martian surface on 02/18/2021. This corresponds to a launch date within four days of our solution and a landing date exactly equal to our solution. It's worth noting, however, that the mission design team targeted a launch window spanning from 07/17/2020 to 08/05/2020 [6], of which our solution falls almost perfectly in the middle. This comparison against the true launch/landing dates of the Mars 2020 mission helps confirm the legitimacy of our results.

V. Conclusions

An optimization problem was formulated to determine the ΔV optimal transfer geometry for interplanetary missions to Mars. By making use of Lambert's algorithm, the search space was reduced to two design variables: Earth departure date and Mars arrival date. By iteratively running Lambert's algorithm across the search space, a "Porkchop" plot was generated that depicted two distinct regions corresponding to Type I and Type II solutions.

Two first-order deterministic methods, gradient descent and Adam, were first implemented to attempt to optimize the objective function. However, these were both found to be sensitive to initial conditions and frequently converged to the locally optimal (Type II) solution. To account for this, the Cross-Entropy algorithm was applied to the optimization problem. By introducing stochastic behavior to the optimization process, this method was more robust to initial conditions and reliably converged to the globally optimal (Type I) solution.

The optimal solution was found to require a transfer cost of $\Delta V = 5.849$ km/s, sufficiently within the limits of modern capabilities. This transfer corresponded to a 07/26/2020 Earth departure date and a 02/18/2021 Mars arrival date. This was compared against the true launch and landing dates of the recent Mars 2020 mission, which helped verify these results. Although a 2020 launch season was chosen for comparison purposes, in future work this research could easily be extended to future launch seasons for preliminary analysis of upcoming missions.

References

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VI. Appendix: Code Repository

All code for this project can be found at the following Github repository: `https://github.com/jgeiser47/AA222_Final_Project`